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A STUDY ON THE STRUCTURE OF A CLASS OF INDEFINITE NON-HYPERBOLIC KAC-MOODY ALGEBRAS QHG_2

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ABSTRACT

Kac-Moody algebras is one of the advanced fields of Mathematical research, which is developing rapidly in recent years due to its interesting connections and applications to many areas in Mathematics and Mathematical Physics like Quantum Physics, Number theory, Combinatorics, Non-linear differential equations etc. A specific class of indefinite non-hyperbolic Kac-Moody Algebras EHG_2 was considered by Uma Maheswari [19] wherein a realization for these algebras as a graded Lie algebra of Kac-Moody type was obtained. The homology modules and the structure of the components of the maximal ideal upto level three were computed. In this paper, a specific class of the family QHG_2 is considered. Using this realization as a graded Lie algebra of Kac-Moody type, the homology modules upto level five are computed. The structure of the components of the maximal ideal upto level four is determined. To compute these we combine the theory of homological techniques and spectral sequences theory.

KEYWORDS: Generalized Cartan Matrix, Kac-Moody Algebra, Finite, Affine, Indefinite, Extended Hyperbolic Algebras, Quasi Hyperbolic, Graded Algebra, Realization, Homology Modules, Spectral Sequences

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1. INTRODUCTION

The theory of Kac-Moody algebra can be classified into Finite, Affine and Indefinite type. Understanding the structure and computing multiplicities of roots explicitly of Kac-Moody algebra is an open and difficult problem. Feingold and Frenkel (1983) computed level 2 root multiplicities for hyperbolic Kac-Moody algebra $HA_1^{(1)}$ and Kang et al (1988) computed root multiplicities of E_{10} , Kang (1993a,b) has computed root multiplicities for roots upto level 5 for $HA_1^{(1)}$, for the roots upto level 3 for $HA_n^{(1)}$ (1994a). Sthanumoorthy and Uma Maheswari (1996a) have computed root multiplicities of roots for a particular class of extended-hyperbolic Kac-Moody algebra $EHA_1^{(1)}$ and again considered the same generally in Sthanumoorthy et al (2004). This class of extended-hyperbolic Kac-Moody algebra was defined in Sthanumoorthy and Uma Maheswari (1996b). Sthanumoorthy and Uma Maheswari (2012) have computed root multiplicities upto level 3 for $EHA_1^{(1)}$ and $EHA_2^{(2)}$. Another class of indefinite non-hyperbolic Kac-Moody algebra called quasi-hyperbolic was introduced by Uma Maheswari (2014).

In this paper we consider the class of quasi-hyperbolic Kac-Moody algebras QHG_2 associated with the GCM $\begin{bmatrix} 2 & -3 & -c \\ -1 & 2 & -a \\ -d & -b & 2 \end{bmatrix}$ where atleast one of ab > 4 or cd > 4. We first give a realization for QHG_2 as a graded Lie algebra of Kac-Moody type and then using the homological techniques developed by Kang and others [14],[16], [17], [18] &[19], we compute the homology modules of QHG_2 upto level 5 and the structure of the components of the maximal ideal upto level 4.

2. PRELIMINARIES

We first recall some results on the general construction of graded Lie algebras of Kac – Moody type (Benkart et al, 1993).

Notations Used

G: Lie algebra over a field of characteristic zero

V, V' : two G – modules.

$\psi : V' \otimes V \rightarrow G$ a G – module homomorphism

$G_0 = G, G_{-1} = V, G_1 = V'$

$G_+ = \sum_{n\geq 1} G_n$ (resp. $G_- = \sum_{n\geq 1} G_{-n}$) – the free Lie algebra generated by V' (resp. V)

G_n (resp G_{-n}) for n > 1 – the space of all products of n vectors from V' (resp. V)

K - An algebraically closed field of characteristic zero.

Now $G = \sum_{n=-\infty}^{\infty} G_n$ is given a Lie algebra structure by defining the Lie bracket [,] as follows:

For a, b ∈ G, $v \in V$, w ∈ V' define $[a,v] = a.v = -[v,a]$ *and* $[a,w] = a.w = -[w,a]$.

Let [a,b] denote the bracket operation in G. For $w \in V', v \in V, [w,v] = \psi(w \otimes v) = -[v,w].$ By extending the bracket operation, $G = \sum_{n\in z} G_n$ becomes a graded Lie algebra which is generated by its local part $G_{-1} + G_{0+} G_1$.

For n≥1 define the subspaces $I_\pm$= {x∈$G_\pm$/[y_1,[...[y_{n-1},x]]...] = 0 for all $y_1,...,y_{n-1} \in G_{\mp 1}\}$. *Set* $I_+ = \sum_{n>1} I_n, I_- = \sum_{n>1} I_{-n.}$

We see that I_+ and I_- are ideals of G and the ideal is the largest graded ideal of G trivially intersecting $G_{-1} + G_0 + G_1$.

For n >1, define $L_{\pm n} = G_{\pm n} / I_\pm$. Let $L = L(G,V,V',\psi) = G_- / I_- \oplus G_0 \oplus G_+ / I_+ = ... \oplus L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 \oplus ...$, where $L_0 = G_0, L_1 = G_1, L_{-1} = G_{-1}$. Then $L = \oplus_{n\in Z} L_n$ is a graded Lie algebra generated by its local part $V \oplus G \oplus V'$ and L = G/I.

Let g = g (A) be a symmetrizable Kac – Moody algebra over K with $A=(a_{ij})_{i,j=1}^{n}$ having rank *l*. Decompose A = DB (where D = diag $(\in_1,...,\in_n)$ and $B=(b_{ij})_{i,j=1}^{n}$, with the realization $(h,\Pi^{\vee}=\{h_1,...,h_n\}$. Let Z be the center of g (A).

Let $H=\langle h_1,...,h_l\rangle$ be a maximal subset of $\Pi^{\vee}$ independent of Z in h. Let $Z_1,\ldots,Z_{n-1}$ denote a basis for Z and d_1, d_{n-1} be linearly independent of Z in $H^{\perp}$ so that $(d_i,z_j)=\delta_{ij}$. Let $V=V_1\oplus V_2\oplus...\oplus V_m$, for some integer m ≥ n-1 each V_i being a faithful, irreducible highest weight module of g of highest weight λ_i where λ_i's are chosen such that $(\lambda_i(z_j))_{i,j=1}^{n-1}$ is non singular. Let $V^*=V_1^*\oplus V_2^*\oplus...\oplus V_m^*$ be the finite dual of V. Let $g^e=g\oplus Kc_{n-l+1}\oplus...\oplus Kc_m$ where the elements c_{n-l+1}, cm act centrally in g^e. Extend the above action of g on v to g_e by letting each c_i act trivially. Now using the basis elements z_i and d_i, we can build an orthonormal basis $y_1, \ldots, y_{2n-1}$ for h and from this we can extract an orthonormal basis for H and this basis $y_1, \ldots, y_{2n-1}$ and root vectors will form a basis $\{x_j\}_{j\in J}$ of g. Then the dual basis x_j^* of g consists of elements y_1, $y2n_{-1}$ and root vectors. Let us assume $(\lambda_i,\lambda_i)\neq 0$ for all i. For $w^*=w_1^*+...+w_m^*\in V^*$ and $v=v_1+...+v_m\in V$ where $w_i^*\in V_i^*, v_i\in V_i$, define $\psi(w^*\otimes v)=\sum_{i=1}^{m}\frac{-2}{(\lambda_i,\lambda_i)}\sum_{j\in J}\langle w_i^*\,|\,x_jv_i\rangle x_j^*+\sum_{i=n-l+1}^{m}\langle w_i^*\,|\,v_i\rangle c_i$ where $\langle\cdot|\cdot\rangle$ denotes the usual pairing $(\langle w_i^*\,|\,v_i\rangle=w_i^*(v_i))$ between V_i^* and V_i. Form the graded Lie algebra L = L(g^e, V, V^*, ψ) .

Theorem 2.1[1]

L is a Z^{n+m} –graded algebra.

Setting $\alpha_{n+i}=-\lambda_i$ for i =1, m, form the matrix $C=(\langle\alpha_i,\alpha_j\rangle)_{i,j=1}^{n+m}$, where $\langle\alpha_i,\alpha_j\rangle=2(\alpha_i,\alpha_j)/(\alpha_i,\alpha_i)$.

Let $\tilde{A}$ (C) be the free Lie algebra on generators E_i, F_i, H_i, i = 1, m and I(C) be the ideal generated by the homogeneous elements

$[H_i,H_j],[H_i,E_j]-\langle\alpha_j,\alpha_i\rangle E_j\,[H_i,F_j]+\langle\alpha_j,\alpha_i\rangle F_j$ And $[E_i,F_j]-\delta_{ij}H_i$. Let $\tilde{A}$ (C) = $\tilde{A}$ (C) / I(C).

Theorem 2.2[1]

Let $\phi: A(C)\to L$ be the Lie algebra homomorphism sending $E_i\to e_i, F_i\to f_i, H_i\to h_i$. Then ϕ has kernel as I(C) and I(C) is the largest graded ideal of A(C) trivially intersecting the span of $H_1,\ldots, H_{n+m}$. Also $\phi: A(C)/I(C)\to L$ is an isomorphism.

Proposition 2.3[1]

The matrix C has rank 2n – *l* and C is symmetrizable.

We now recall the definition of homology of Lie algebra (Garland and Lepowsky, 1976) and Hochschild-Serre spectral sequence (Kang, 1993a).

Let G be a Lie-algebra and V and module over G. Define the space $C_q(G,V)$ for q > 0 of Q – dimensional chains of the Lie algebra G with coefficients in V to be $\wedge^q(G)\otimes V$. The differential $dq = Cq(G,V) \to C_{q-1}(G,V)$ is defined to be $d_q(g_1 \wedge \ldots \wedge g_q \otimes v) = \sum_{1\le s\le t\le q} (-1)^{s+t-1}([g_s, g_t]) \wedge g_1 \wedge \ldots \hat{g}_s \wedge \ldots \wedge \hat{g}_t \wedge \ldots \wedge g_q) \otimes v + \sum_{1\le s\le q} (-1)^s (g_1 \wedge \ldots \wedge \hat{g}_s \wedge \ldots \wedge g_q) \otimes g_s.v,$

For $v \in V,\ g_1, g_2, ..., g_q \in G.$ For q < 0, define $C_q(G,V) = 0$ and $d_q = 0$. Then $d_q \circ d_{q+1} = 0.$ The homology of the complex $(C,d) = \{C_q(G,V), dq\}$ is called the homology of the Lie algebra G with coefficients in V and is denoted by $H_q(G,V)$. If V = C, we simply write $H_q(G)$ for $H_q(G,C)$. Assume now that G, V, C_q (G,V) are completely reducible modules in the category O over a Kac-Moody algebra g(A) with d_q having g(A)-module homomorphisms. Let I be ideal of G and L = G/I. Define a filtration $\{K_p=K_pC\}$ of the complex {C, d} by $K_pC_{p+q} = \{g_1 \wedge g_2 \wedge ... \wedge g_{p+q} \otimes v \mid g_i \in I \ \ for\ p+1 \le i \le p+q\}.$

This gives rise to a spectral sequence $\{E^r_{p,q}, d_r : E^r_{p,q} \to E^r_{p-1,q+r-1}\}$ such that $E^2_{p,q} \cong H_p(L, H_q(I,V))$, where $E^r_{p,q}$'s are determined by $E^{r+1}_{p,q} = Ker(d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}) / \mathrm{Im}(d_r : E^r_{p+r,q-r+1} \to E^r_{p,q})$ with boundary homomorphisms $d_{r+1} : E^r_{p,q} \to E^r_{p-r-1,q+r}$. The modules $E^r_{p,q}$ become stable for r > max(p,q +1) for each (p,q) and the stable module is denoted by $E^\infty_{p,q}$. The spectral sequence $\{E^r_{p,q}, d_r\}$ converges to $H_n(G,V)$ in the following sense: $H_n(G,V) = \oplus_{p+q=n} E^\infty_{p,q}$.

Then we have the following Hochschild-Serre five term exact sequences (Kang et al, 1994):

$$H_2(G,V) \to H_2(L, H_0(I,V) \to H_0(L, H_1(I,V)) \to H_1(G,V) \to H_1(L, H_0(I,V)) \to 0.$$

Now consider $G = \underset{n\ge1}{\oplus} G_n$ be the free Lie algebra generated by the subspace G_1 and $I = \underset{n\ge m}{\oplus} I_n$ be the graded ideal of G generated by the subspace I_m for $m \ge 2.$ Consider the quotient algebra L = G/I. Then $L = \underset{n\ge1}{\oplus} L_n$ is also a graded Lie algebra generated by the subspace $L_1 = G_1$. Let J = I / [I, I]. J is an L-module via adjoint action generated by the subspace J_m.

As vector spaces, $J_n \cong I_n$ for $m \le n < 2m$. Suppose that I_m and G_1 are modules over a Kac-Moody algebra g (A). Then G_n has a g(A)-module structure such that $x\cdot[v,w] = [x\cdot v, w] + [v, x\cdot w]$ for $x \in g(A), v \in G, w \in G_{n-1}; I_n$ also has a similar module structure. We also have the induced module structure of the homogeneous subspaces L_n, J_n.

Then we have the following theorem proved in Kang (1993a).

Theorem 2.4[8]

There is an isomorphism of g(A) – modules $H_j(L,J) \cong H_{j+2}(L),$ for $J \ge 1.$ In particular $I_{m+1} \cong (G_1 \otimes I_m) / H_3(L)_{m+1}$

Now, for arbitrary $j \ge m,$ set $I^{(j)} = \sum_{n\ge j} I_n;$ then $I_{(j)}$ is an ideal of G generated by the subspace I_j. We consider the quotient algebra $L^{(j)} = G/I^{(j)}$. Let $N^{(j)}=I^{(j)}/ I^{(j-1)}$. In this notation $L = L^{(m)}$. Then we have an important relation: $I_{j+1} \cong (G_1 \otimes I_j) / H_3(L^{(j)})_{j+1}$. And, there exists a spectral sequence $\{E^r_{p,q}, d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}\}$ converging to $H_*(L^{(j)})$ such that and $E^e_{p,q} \cong H_p(L^{(i-1)}) \otimes \wedge^q(I_{j-1})$ and $H_3(L^{(j)}) \cong E^\infty_{3,0} \oplus E^\infty_{2,1} \oplus E^\infty_{1,2} \oplus E^\infty_{0,3}$

Lemma 2 5[8]

In the above notation, $H_2(L) \cong \mathrm{I}_m$.

Now we recall the Kostant's formula for symmetrizable Kac-Moody algebras (Liu, 1992):

Let A=$(a_{ij})_{i,j=1}^{n}$ be a symmetrizable GCM. Let $\Delta \subset \mathfrak{h}^*$, Δ^+, Δ^- denote the root system of g(A), positive and negative roots, respectively, of g(A). We have the triangular decomposition: $g(A) = n^- \oplus h \oplus n^+$, where $n^{\pm} = \underset{\alpha\in\Delta^{\pm}}{\oplus} g_\alpha$

Let S={1,…,s} be a subset of N = {1,…,n} and g_s be the subalgebra of g(A) generated by the elements e_i,f_i, i = 1, …, s and h. Let Δ_s^+ denote the set of positive roots generated by $\alpha_1,...,\alpha_s$ *and* $\Delta_s^- = -\Delta_s^+$. Then g_s has the corresponding triangular decomposition: $g_s = n_s^- \oplus h \oplus n_s^+$, where $n_s^+ = \underset{\alpha\in\Delta_s^{\pm}}{\oplus} g_\alpha$ and $\Delta_s = \Delta_s^+ \cup \Delta_s^-$ is the root system of g_s. Let $\Delta^{\pm}(S) = \Delta^{\pm} \setminus \Delta_s^{\pm}$ and $n^{\pm}(S) = \underset{\alpha\in\Delta^{\pm}(s)}{\oplus} g_\alpha$ Then $g(A) = n^-(S) \oplus g_s \oplus n^+(S)$. Let $W(S) = \{w \in W \mid w\Delta^- \cap \Delta^+ \subset \Delta^+(S)\}$.

For $\lambda \in h^*$ denote by $\tilde{V}(\lambda)$, the irreducible highest weight module over g(A) and $V(\lambda)$ the irreducible highest weight module over g_s.

Theorem 2.6[5]

(Kostant's formula) $H_j(n^-(S), \tilde{V}(\lambda)) \cong \underset{\substack{w\in W(s)\\ l(w)=j}}{\oplus} V(w(\lambda+\rho)-\rho)$.

Lemma 2. 7[8]

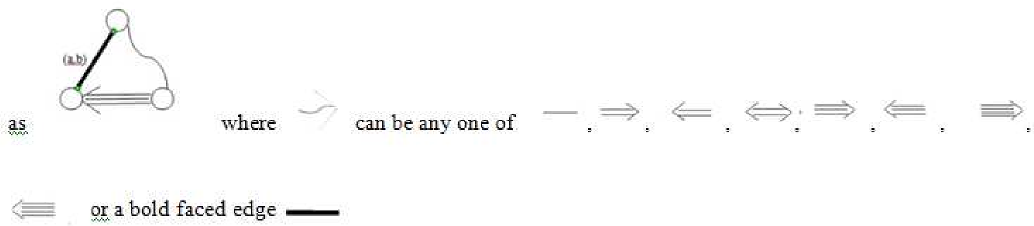
Suppose $w = w'\ r_j$ and $l(w) = l(w') + 1$. Then $w \in W(S)$ if and only if $w' \in W(S)$ and $w'(\alpha_j) \in \Delta^+(S)$.

Definition 2. 8[16]

Let $A = (a_{ij})_{i,j=1}^{n}$ be an indecomposable GCM of indefinite type. The associated Dynkin diagram S (A) to be Quasi Hyperbolic (QH) type if S (A) has a proper connected sub diagram of hyperbolic type with n-1 vertices. The GCM A is of QH type if S (A) is of QH type. Then the Kac-Moody algebra is of QH type.

3. RELIZATION FOR QHG_2

Let us denote by QHG_2, the class of quasi-hyperbolic Kac-Moody algebras whose associated GCM is of the form $\begin{bmatrix} 2 & -3 & -c \\ -1 & 2 & -a \\ -d & -b & 2 \end{bmatrix}$, where atleast one of ab > 4 or cd > 4 that is, the class of all 3 x 3 GCM of quasi – hyperbolic type obtained from the algebra G_2 associated with the GCM $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. Here a, b, c, d, $\in \mathbf{Z}^+$. The associated Dynkin diagram is represented



Consider the Kac-Moody algebra associated with the GCM $A=\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$.

Let $(h,\Pi,\Pi^\vee)$ be the realization of A with $\Pi=\{\alpha_1,\alpha_2\}$ and $\Pi^\vee=\{\alpha_1^\vee,\alpha_2^\vee\}$. Then we have the following bilinear relations $(\alpha_1,\alpha_1)=2/3,\ (\alpha_1,\alpha_2)=-1,\ (\alpha_2,\alpha_2)=2,$

Let $\alpha_3^{'}$ be the element in h* such that $\alpha_3^{'}(\alpha_1^\vee)=0, \alpha_3^{'}(\alpha_2^\vee)=1.$ Let us define $\lambda=-a\alpha_1-\left(\frac{2a+c}{3}\right)\alpha_2+\frac{2(2a+3)}{3}\alpha_3^{'}$.

Set α_3 = -λ. hen $\langle\alpha_3,\alpha_1\rangle=-d$, $\langle\alpha_1,\alpha_3\rangle=-c$, $\langle\alpha_3,\alpha_2\rangle=-b;$ $\langle\alpha_2,\alpha_3\rangle=-a$, $\langle\alpha_3,\alpha_3\rangle=2;$ $\langle\alpha_2,\alpha_1\rangle=-1$, $\langle\alpha_1,\alpha_2\rangle=-3$

Form the matrix $C=\left(\langle\alpha_i,\alpha_j\rangle\right)_{i,j=1}^{3}$.Then $C=\begin{bmatrix} 2 & -3 & -c \\ -1 & 2 & -a \\ -d & -b & 2 \end{bmatrix}$ is a symmetrizable GCM of quasi hyperbolic type.

Let V be the integrable highest weight irreducible module over G with the highest weight λ as defined. Let V* be the contragradient of V and ψ be the mapping as defined earlier. Let G be the Kac-Moody algebra associated with the GCM $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. Form the graded Lie algebra L (G, V, V*, ψ). Then $L\cong g(C)$ and L is a symmetrizable Kac-Moody algebra of quasi-hyperbolic type associated with the GCM C.

Next we compute the homology modules of the Kac-Moody algebra for QHG_2.We note here that, from the realization of $L=QHG_2$ as $L=L\_\oplus L_0\oplus L_+=G/I$ and using the involutive automorphism , it suffices to study about the negative part L_ = G_ / I_.

Computation of Homology Modules

Let S = {1, 2} $\subset N=\{1,2,3\}$. Here g_s is the Kac-Moody Lie algebra G_2. Here $\Delta^+(S)=\{k_1\alpha_1+k_2\alpha_2+k_3\alpha_3^{'}\in\Delta^+ \mid k_3\neq 0\}$. Let Δ_s be the root system of g_s. $H_1(L\_)\cong V(-\alpha_3)$. The only element length 1 in W(S) is r_3.

Hence by the Kostant's formula we see that the elements of length 2 in W(S) are r_3r_1 and r_3r_2.We have $r_3r_1\rho-\rho=-\alpha_1-(d+1)\alpha_3$ and $r_3r_2\rho-\rho=-(b+1)\alpha_3-\alpha_2$.

Hence, $H_2(L\_)\cong V(-(d+1))\alpha_3-\alpha_1)\oplus V((-b+1)\alpha_3-\alpha_2)$.

Elements of length 3 in W(S) are $r_3r_2r_3, r_3r_2r_1, r_3r_1r_2$ and $r_3r_1r_3$.

We then have, $r_3r_2r_3\rho - \rho = -(ab+b)\alpha_3 - (a+1)\alpha_2$

$r_3r_2r_1\rho - \rho = -\alpha_1 - 2\alpha_2 - (2b+d+1)\alpha_3$,

$r_3r_1r_2\rho - \rho = -4\alpha_1 - \alpha_2 - (4d+b+1)\alpha_3$, $r_3r_1r_3\rho - \rho = -(1+c)d\alpha_3 - (1+c)\alpha_1$

Hence, $H_3(L_-) \cong V(-(ab+b)\alpha_3 - (a+1)\alpha_2) \oplus V(-(4d+b+1)\alpha_3 - \alpha_2 - 4\alpha_1)$

$\oplus V(-(2b+d+1)\alpha_3 - 2\alpha_2 - \alpha_1) \oplus V(-(1+c)d\alpha_3 - (1+c)\alpha_1)$

Elements of length 4 in W(S) are $r_3r_2r_1r_2$, $r_3r_2r_1r_3$, $r_3r_2r_3r_1$, $r_3r_2r_3r_2$, $r_3r_1r_2r_1$, $r_3r_1r_2r_3$, $r_3r_1r_3r_1$, $r_3r_1r_3r_2$,

$r_3r_2r_1r_2(\rho) - \rho = -(1+4d+4b)\alpha_3 - 4\alpha_2 - 4\alpha_1$

$r_3r_2r_1r_3(\rho) - \rho = -(b(2+a+c)+d(1+c))\alpha_3 - (2+a+c)\alpha_2 - (1+c)\alpha_1$

$r_3r_2r_3r_1(\rho) - \rho = -(2+a(1+d))b\alpha_3 - (2+a(1+d))\alpha_2 - \alpha_1$

$r_3r_2r_3r_2(\rho) - \rho = -((1+b)ab - d)\alpha_3 - (1+b)a\alpha_2$

$r_3r_1r_2r_1(\rho) - \rho = -(1+2b+6d)\alpha_3 - 2\alpha_2 - 6\alpha_1$

$r_3r_1r_2r_3(\rho) - \rho = -(ab+bc+cd+4d+b)\alpha_3 - (1+a)\alpha_2 - (4+3a+c)\alpha_1$

$r_3r_1r_3r_1(\rho) - \rho = -(cd(1+d) - d)\alpha_3 - c(1+d)\alpha_1$

$r_3r_1r_3r_2(\rho) - \rho = -(4d+dc+bcd)\alpha_3 - \alpha_2 - (4+c+bc)\alpha_1$

Hence $H_4(L-) \cong V(-(1+4d+4b)\alpha_3 - 4\alpha_2 - 4\alpha_1) \oplus V(-(b(2+a+c)+d(1+c))\alpha_3 - (2+a+c)\alpha_2 - (1+c)\alpha_1)$

$\oplus V(-(2+a(1+d))b\alpha_3 - (2+a(1+d))\alpha_2 - \alpha_1) \oplus V(-((1+b)ab-d)\alpha_3 - (1+b)a\alpha_2$

$\oplus V(-(1+2b+6d)\alpha_3 - 2\alpha_2 - 6\alpha_1)$

$\oplus V(-(ab+bc+cd+4d+b)\alpha_3 - (1+a)\alpha_2 - (4+3a+c)\alpha_1)$

$\oplus V(-(cd(1+d)-d)\alpha_3 - c(1+d)\alpha_1) \oplus V(-(4d+dc+bcd)\alpha_3 - \alpha_2 - (4+c+bc)\alpha_1)$

Elements of length 5 in W(S) are

$r_3r_2r_1r_2r_1$ r3r2r1r2r3, $r_3r_2r_1r_3r_1$, $r_3r_2r_1r_3r_2$, $r_3r_2r_3r_1r_2$, $r_3r_2r_3r_1r_3$, $r_3r_2r_3r_2r_1$, $r_3r_2r_3r_2r_3$, $r_3r_1r_2r_1r_2$, $r_3r_1r_2r_1r_3$, $r_3r_1r_2r_3r_1$, $r_3r_1r_2r_3r_2$, $r_3r_1r_3r_1r_2$, $r_3r_1r_3r_1r_3$, $r_3r_1r_3r_2r_1$, $r_3r_1r_3r_2r_3$

We have, $r_3r_2r_1r_2r_1(\rho) - \rho = -\alpha_3(1+5b+6d) - 5\alpha_2 - 6\alpha_1$

$r_3r_2r_1r_2r_3 (\rho) - \rho = - \alpha_3 (3ab + 2bc + cd + 4b + 4d) - (4 + c + 3a) \alpha2 - (4 + c + 3a) \alpha1$

$r_3r_2r_1r_3r_1 (\rho) - \rho = - \alpha_3 (cd^2 + 2cd + ad - d + a + c + 1) - (1 + a + ad + c + cd) \alpha2 - c (1 + d) \alpha1$

$r_3r_2r_1r_3r_2(\rho) - \rho = - \alpha_3[b(a + ab + 4 + c + bc) + d(4 + c + bc) - b] - (a + ab + 4 + c + bc)\alpha_2 - (4 + c + bc)\alpha_1$

$r_3r_2r_3r_1r_2 (\rho) - \rho = - \alpha_3 (- b - 4d + ab + ab^2 - 4abd - 4b) - a (1 + b + 4d)\alpha_2 - 4\alpha_1$

$r_3r_2r_3r_1r_3 (\rho) - \rho = - \alpha_3 (1 - d (1 + a) + b (2 + a - ad - a^2d + d + ad) - (2 + a - ad (1 + a))\alpha_2 - (1 + a)\alpha_1$

$r_3r_2r_3r_2r_1 (\rho) - \rho = - \alpha_3 (ab(1 + 2b + d) - 2b) - a(1 + 2b + d)\alpha_2 - \alpha_1$

$r_3r_2r_3r_2r_3 (\rho) - \rho = - \alpha_3 (1 - b + a + ab) - a\alpha_2$

$r_3r_1r_2r_1r_2 (\rho) - \rho = - \alpha_3 (1 + 4b + 9d) - 4\alpha_2 - 9\alpha_1$

$r_3r_1r_2r_1r_3 (\rho) - \rho = - \alpha_3 (b (2 + c + a) + 3(2 + c + a) d) - (2 + c + a) \alpha2 - (2 + c + a) 3\alpha1$

$r_3r_1r_2r_3r_1 (\rho) - \rho = - \alpha_3 (b (2 + a + ad) + d^2 + 6 + 3a + 3ad) - (2 + a + ad) \alpha2 - (1 + d + 3(2 + a +ad)) \alpha1$

$r_3r_1r_2r_3r_2 (\rho) - \rho = - \alpha_3 (ba(1 + b) - d(1 + (1 + b)(c - 3a) - b)) - a(1 + b)\alpha_2 - (1 + (1 + b)(c - 3a)\alpha_1$

$r_3r_1r_3r_1r_2 (\rho) - \rho = - \alpha_3 ((4d + b) d - 3d) - \alpha_2 - (1 + 4d + b) \alpha1$

$r_3r_1r_3r_1r_3 (\rho) - \rho = - \alpha_3 (c^2d^2 + cd^2 - 2cd - d - 1) - (cd + c^2d - c) \alpha1$

$r_3r_1r_3r_2r_1 (\rho) - \rho = - \alpha_3 [b (2 + 2b) + (6d + 6b + d^2)] - (2 + 2b) \alpha2 - (7 + 6b + d) \alpha1$

$r_3r_1r_3r_2r_3 (\rho) - \rho = - \alpha_3 [1 + d (bc + abc - 2 - a] - (1 + a) \alpha2 - (bc + abc - 2 - a) \alpha1$

Hence, $H_5 (L\_) \cong V (- \alpha_3 (1 + 5b + 6d) - 5\alpha_2 - 6\alpha_1) \oplus V (- \alpha_3 (3ab + 2bc + cd + 4b + 4d) - (4 + c + 3a) \alpha2 - (4 + c + 3a)\alpha_1)$

$\oplus V (- \alpha_3 (cd^2 + 2cd + ad - d + a + c + 1) - (1 + a + ad + c + cd) \alpha2 - c (1 + d) \alpha1)$

$\oplus V(- \alpha_3[b(a + ab + 4 + c + bc) + d(4 + c + bc) - b] - (a + ab + 4 + c + bc)\alpha_2 - (4 + c + bc)\alpha_1)$

$\oplus V (- \alpha_3 (- b - 4d + ab + ab^2 - 4abd - 4b) - a (1 + b + 4d) \alpha2 - 4\alpha_1)$

$\oplus V (- \alpha_3 (1 - d (1 + a) + b (2 + a - ad - a^2d + d + ad) - (2 + a - ad (1 + a)) \alpha2 - (1 + a) \alpha1)$

$\oplus V (- \alpha_3 (ab (1 + 2b + d) - 2b) - a (1 + 2b + d) \alpha2 - \alpha_1) \oplus V (- \alpha_3 (1 - b + a + ab) - a\alpha_2)$

$\oplus V (- \alpha_3 (1 + 4b + 9d) - 4\alpha_2 - 9\alpha_1) \oplus V (- \alpha_3 (b (2 + c + a) + 3(2 + c + a) d) - (2 + c + a) \alpha2 - (2 + c + a) 3\alpha1)$

$\oplus V (- \alpha_3 (b (2 + a + ad) + d^2 + 6 + 3a + 3ad) - (2 + a + ad) \alpha2 - (1 + d + 3(2 + a +ad)) \alpha1)$

$\oplus V (- \alpha_3 (ba (1 + b) - d (1 + (1 + b) (c - 3a) - b)) - a (1 + b) \alpha2 - (1 + (1 + b) (c - 3a) \alpha1))$

$\oplus V (- \alpha_3 ((4d + b) d - 3d) - \alpha_2 - (1 + 4d + b) \alpha1) \oplus V (- \alpha_3 (c^2d^2 + cd^2 - 2cd - d - 1) - (cd + c^2d - c) \alpha1)$

$\oplus V (- \alpha_3 [b (2 + 2b) + (6d + 6b + d^2)] - (2 + 2b) \alpha2 - (7 + 6b + d) \alpha1)$

$\oplus V (- \alpha_3 [1 + d (bc + abc - 2 - a] - (1 + a) \alpha2 - (bc + abc - 2 - a) \alpha1)$

Similarly, we can compute the other homology modules $H_6(L_-)$, $H_7(L_-)$, $H_8(L_-)$ etc.

4. STRUCTURE OF THE MAXIMAL IDEAL IN QHG_2

In the this section, using the homological approach together with the representation theory of Kac-Moody algebra we will determine some of the boundary homomorphisms and deduce some new structural information on maximal ideals in QHG_2. We know that the ideal I_- of G_- is generated by the homological subspace I_{-2}, and hence we may write $I_- = I_-^{(2)}$.

Similarly, for $j \geq 2$, we write $I_-^{(j)} = \sum_{n \geq j} I_{-n}, L_-^{(j)} = G / I_-^{(j)}$ and $N_-^{(j)} = I_-^{(j)} / I_-^{(j+1)}$. By the homological theory, we have in general that $I_{-(j+1)} \cong (V \otimes I_{-j}) / H_3(L_-^{(j)})_{-(j+1)}$ *for* $j \geq 2$

Since G_ is free and I_ is generated by the subspace I_{-2} from the Hochschild –Serre five term exact sequence and using Lemma proved earlier, we see that $I_{-2} \cong H_2(L_-)$; $H_2(L_-) \cong V(-(d+1)\alpha_3 - \alpha_1) \oplus V(-(b+1)\alpha_3 - \alpha_2)$.

Hence, $I_{-2} \cong H_2(L_-) \cong V(-(d+1)\alpha_3 - \alpha_1) \oplus V(-(b+1)\alpha_3 - \alpha_2)$

When j = 2, $L_-^{(2)}$ coincides with the subspace $\eta^-(S)$ for S = {1, 2} and we can computer $H_3(L_-^{(2)})$ using the Kostant formula.

$$H_3(L_-) \cong V(-(ab+b)\alpha_3 - (a+1)\alpha_2) \oplus V(-(4d+b+1)\alpha_3 - \alpha_2 - 4\alpha_1)$$

$$\oplus V(-(2b+d+1)\alpha_3 - 2\alpha_2 - \alpha_1) \oplus V(-(1+c)d\alpha_3 - (1+c)\alpha_1)$$

To find $H_3(L_-^{(2)})_{-3}$

Case (1) If ab > 4, cd > 4, 3ad = bc, $H_3(L_-^{(2)})_{-3} = 0$, Hence $I_{-3} \cong V \otimes I_{-2}$

Case (2) If ab > 4, cd ≤ 4

$H_3(L_-^{(2)})_{-3} \cong V(-(1+c)d\alpha_3 - (1+c)\alpha_1)$ if c=2,d=1,3a=2b

$\cong 0$ Otherwise

Hence $I_{-3} \cong (V \otimes I_{-2}) / V(-(1+c)d\alpha_3 - (1+c)\alpha_1)$

$\cong 0$ Otherwise

Case (3) If ab ≤ 4, cd > 4,

$H_3(L_-^{(2)})_{-3} \cong V(-b(a+1)\alpha_3 - (a+1)\alpha_2)$ if a=2,b=1,6d=c

$\cong 0$ Otherwise. Hence $I_{-3} \cong (V \otimes I_{-2}) / V(-b(a+1)\alpha_3 - (a+1)\alpha_2)$

To determine the structure of I_{-4}: To determine the structure of I_{-4}, we need to determine the structure of $H_3(L_-^{(3)})_{-4}$. We consider the following short exact sequence,$0 \to N_-^{(2)} \to L_-^{(3)} \to L_-^{(2)} \to 0$ and the corresponding spectral

sequence $\{E^r_{p,q}\}$ converging to $H_*(L^{(3)}_-)$ such that $E^2_{p,q} \cong H_p(L^{(2)}_-) \otimes \Lambda^q(I_{-2})$.

We will compute $H_3(L^{(3)}_-)_{-4}$ from this sequence, Let us start with the sequence $0 \to E^2_{2,0} \xrightarrow{d_2} E^2_{0,1} \to 0.$ Note that $H_1(L^{(3)}_-) \cong L^{(3)}_-/[L^{(3)}_-, L^{(3)}_-] \cong L_{-1} = V.$ Since the spectral sequence converges to $H_*(L^{(3)}_-)$, we have $H_1(L^{(3)}_-) \cong E^\infty_{1,0} \oplus E^\infty_{0,1}.$ But $E^\infty_{1,0} = E^2_{1,0} \cong H_1(L^{(2)}_-) \cong L^{(2)}_1/[L^{(2)}_-, L^{(2)}_-] \cong L_{-1} = V,$ which implies $E^\infty_{0,1} = E^3_{0,1} = 0.$ Hence the homomorphism d_2 is surjective. Since $E^2_{2,0} \cong I_{-2}$ and $E^2_{0,1} \cong I_{-2}, d_2$ must be an isomorphism. Thus $E^3_{2,0} = 0,$ and hence $E^\infty_{2,0} = 0.$

Now consider the following sequence, $0 \to E^2_{3,0} \xrightarrow{d_2} E^2_{1,1} \to 0.$ By the Kostant formula, we have

$$E^2_{3,0} \cong H_3(L^{(2)}_-) \cong \{V(\text{-b(a}+1)\alpha_3 - (a+1)\alpha_2) \oplus V(-(2b+d+1)\alpha_3 - 2\alpha_2 - \alpha_1)$$

$$\oplus V(-4d+b+1)\alpha_3 - \alpha_2 - 4\alpha_1) \oplus V(-(1+c)d\alpha_3 - (1+c)\alpha_1)\}$$

and $E^2_{1,1} \cong H_1(L^{(2)}_-) \otimes I_{-2} \cong V \otimes I_{-2}$ .Since $V \otimes I_{-2}$ is a direct sum of irreducible highest weight modules over QHG_2 of level 3, by comparing the levels of both terms, we see that d_2: $E^2_{3,0} \to E^2_{1,1}$ is trivial. So $E^3_{3,0} = E^2_{3,0}$, and $E^\infty_{1,1} = E^3_{1,1} = E^2_{1,1} \cong V \otimes I_{-2}.$ since $I^{(3)}_-$ is generated by I_{-3}, by using Lemma 5 and Theorem 4 give $H_2(L^{(3)}_-) \cong I_{-3} = V \otimes I_{-2}.$ But we have $H_2(L^{(3)}_-) \cong E^\infty_{2,0} \oplus E^\infty_{1,1} \oplus E^\infty_{0,2}.$ it follows that $E^\infty_{0,2} = E^4_{0,2} = 0.$ hence we conclude either $E^3_{0,2} = 0$ or the homomorphism $d_3 : E^3_{3,0} \to E^3_{0,2}$ is surjective.

Assume first that $E^3_{0,2} = 0$. This implies that $d_3 : E^3_{3,0} \to E^3_{0,2}$ is trivial and that the homomorphism $d_2 : E^2_{2,1} \to E^2_{0,2}$ is surjective in the sequence $0 \to E^2_{4,0} \xrightarrow{d_2} E^2_{2,1} \xrightarrow{d_2} E^2_{0,2} \to 0.$

Thus $E^\infty_{3,0} = E^4_{3,0} = \ker(d_3 : E^3_{3,0} \to E^3_{0,2})/\mathrm{Im}(d_3 : 0 \to E^3_{3,0})$

$$= E^3_{3,0} = E^2_{3,0} \cong \{V(\text{-b(a}+1)\alpha_3 - (a+1)\alpha_2) \oplus V(-(2b+d+1)\alpha_3 - 2\alpha_2 - \alpha_1)$$

$$\oplus V(-4d+b+1)\alpha_3 - \alpha_2 - 4\alpha_1) \oplus V(-(1+c)d\alpha_3 - (1+c)\alpha_1)\}.$$

By comparing levels, we see that $d_2 : E^2_{4,0} \to E^2_{2,1}$ must be trivial. Note that $E^2_{0,2} \cong \Lambda^2(I_{-2}).$ Therefore $E^3_{4,0} = E^2_{4,0}$ and

$E^\infty_{2,1} = E^3_{2,1} = \mathrm{Ker}(d_2 : E^2_{2,1} \to E^2_{0,2})/\mathrm{Im}(d_2 : E^2_{4,0} \to E^2_{2,1}) \cong \mathrm{Ker}\ (d_2 : E^2_{2,1} \to E^2_{0,2}).$ Since $d_2 : E^2_{2,1} \to E^2_{0,2}$ is surjective, we have $\Lambda^2(I_{-2}) \cong E^2_{0,2} \cong E^2_{2,1}/\mathrm{Ker}\ d_2 \cong I_{-2} \otimes I_{-2}/\mathrm{Ker}\ d_2$. Therefore $\mathrm{Ker}\, d_2 \cong S^2(I_{-2}).$ Hence $E^\infty_{2,1} \cong S^2(I_{-2}).$

If $E^3_{0,2}$ is nonzero and $d_3 : E^3_{3,0} \to E^3_{0,2}$ is surjective, then since $E^3_{3,0} = E^2_{3,0}$ is irreducible, $d_3 : E^3_{3,0} \to E^3_{0,2}$ is an isomorphism. Thus $E^\infty_{3,0} = E^4_{3,0} = 0$ and

$\{V(\text{-b(a}+1)\alpha_3-(a+1)\alpha_2)\oplus V(-(2b+d+1)\alpha_3-2\alpha_2-\alpha_1)\oplus V(-4d+b+1)\alpha_3-\alpha_2-4\alpha_1)\oplus V(-(1+c)d\alpha_3-(1+c)\alpha_1)\}$
$\cong E^3_{3,0}\cong E^3_{0,2}\cong E^2_{0,2}/\mathrm{Im}(d_2:E^2_{2,1}\to E^2_{0,2})$

$\cong\Lambda^2(I_{-2})/\mathrm{Im}(d_2:E^2_{2,1}\to E^2_{0,2})$.

Since all the modules involved here are completely reducible over QHG$_2$, we have

$$\mathrm{Im}(d_2:E^2_{2,1}\to E^2_{0,2})\cong\Lambda^2(I_{-2})/\{V(\text{-b(a}+1)\alpha_3-(a+1)\alpha_2)\oplus V(-(2b+d+1)\alpha_3-2\alpha_2-\alpha_1)$$

$$\oplus V(-4d+b+1)\alpha_3-\alpha_2-4\alpha_1)\oplus V(-(1+c)d\alpha_3-(1+c)\alpha_1)\}$$

We have seen the homomorphism $d_2:E^2_{4,0}\to E^2_{2,1}$ is trivial. Thus

$E^\infty_{2,1}=E^3_{2,1}=Ker(d_2:E^2_{2,1}\to E^2_{0,2})/\mathrm{Im}(d_2:E^2_{4,0}\to E^2_{2,1})=Ker(d_2:E^2_{2,1}\to E^2_{0,2})$. Since

$\mathrm{Im}\,d_2\cong\Lambda^2(I_{-2})/\{V(\text{-b(a}+1)\alpha_3-(a+1)\alpha_2)\oplus V(-(2b+d+1)\alpha_3-2\alpha_2-\alpha_1)$

$\oplus V(-4d+b+1)\alpha_3-\alpha_2-4\alpha_1)\oplus V(-(1+c)d\alpha_3-(1+c)\alpha_1)\}\cong E^2_{2,1}/\mathrm{Ker}\ d_2\cong I_{-2}\otimes I_{-2}/\mathrm{Ker}\ d_2$,we have,

Ker $d_2\cong S^2(I_{-2})\oplus\{V(\text{-b(a}+1)\alpha_3-(a+1)\alpha_2)\oplus V(-(2b+d+1)\alpha_3-2\alpha_2-\alpha_1)$

$\oplus V(-4d+b+1)\alpha_3-\alpha_2-4\alpha_1)\oplus V(-(1+c)d\alpha_3-(1+c)\alpha_1)\}$

Therefore in either case, we have

$$E^\infty_{3,0}\oplus E^\infty_{2,1}\cong S^2(I_{-2})\oplus\{V(\text{-b(a}+1)\alpha_3-(a+1)\alpha_2)\oplus V(-(2b+d+1)\alpha_3-2\alpha_2-\alpha_1)$$

$$\oplus V(-4d+b+1)\alpha_3-\alpha_2-4\alpha_1)\oplus V(-(1+c)d\alpha_3-(1+c)\alpha_1)\}$$

Now consider the sequence $0\to E^2_{5,0}\xrightarrow{d_2}E^2_{3,1}\to 0.$ by comparing levels, we see that the homomorphism $d_2:E^2_{3,1}\to E^2_{1,2}$ is trivial. Thus $E^3_{1,2}=E^2_{1,2}\cong V\otimes\Lambda^2(I_{-2})$. Again by comparing the levels of the terms in the sequence $0\to E^3_{4,0}\xrightarrow{d_3}E^3_{1,2}\to 0$, we conclude that $d_3=0$. Therefore $E^\infty_{1,2}=E^4_{1,2}=E^2_{1,2}\cong V\otimes\Lambda^2(I_{-2})$.

Finally, since $E^\infty_{0,3}$ is a sub module of $E^2_{0,3}\cong\Lambda^3(I_{-2})$, we see that

$$H_3(L^{(3)}_-)\cong\{V(\text{-b(a}+1)\alpha_3-(a+1)\alpha_2)\oplus V(-(2b+d+1)\alpha_3-2\alpha_2-\alpha_1)$$

$$\oplus V(-4d+b+1)\alpha_3-\alpha_2-4\alpha_1)\oplus V(-(1+c)d\alpha_3-(1+c)\alpha_1)\}\oplus S^2(I_{-2})\oplus(V\otimes\Lambda^2(I_{-2}))\oplus M,$$

Where M is a direct sum of level 6 irreducible representations of QHG$_2$

Case (1) If ab > 4, cd > 4, $H_3(L^{(3)}_-)_{-4}=0, I_{-4}\cong(V\otimes I_{-3})/S^2(I_{-2})$

Case (2) If ab > 4, cd ≤ 4, $H_3(L^{(3)}_-)_{-4}\cong V(-(1+c)d\alpha_3-(1+c)\alpha_1)$ if $c=1, d=2, 6a=b.$

$\cong 0$, otherwise

$$I_{-4} \cong (V \otimes I_{-3}) / \ V(-(1+c)d\alpha_3 - (1+c)\alpha_1) \otimes S^2(I_{-2})$$

Case (3) If ab ≤ 4, cd > 4, $H_3(L_-^{(3)})_{-4} \cong V(-b(1+a)\alpha_3 - (1+a)\alpha_2)$ if $a = 1, b = 2, 3d = 2c$.

$\cong 0$, otherwise.

$$I_{-4} \cong (V \otimes I_{-3}) / \ V(-b(1+a)\alpha_3 - (1+a)\alpha_2) \otimes S^2(I_{-2})$$

From the above equations we get the structure of the components of the maximal ideal I_- (upto level 3) in the extended – hyperbolic Kac-Moody algebra QHG_2. Thus we have proved the following Theorem.

Theorem 10

With the usual notations, let $L = \oplus_{n \in Z} L_n$ be the realization of QHG_2 associated with the GCM $\begin{bmatrix} 2 & -3 & -c \\ -1 & 2 & -a \\ -d & -b & 2 \end{bmatrix}$

where atleast one of ab > 4 or cd > 4.Then we have following:

- $I_{-2} \cong V(-(d+1)\alpha_3 - \alpha_1) \oplus V(-(b+1)\alpha_3 - \alpha_2)$
- $I_{-3} \cong \begin{cases} V \otimes I_{-2}, \ \ if \ \ ab > 4, cd > 4 \\ (V \otimes I_{-2}) / V(-(1+c)d\alpha_3 - (1+c)\alpha_1 \ \ if \ \ c = 2, d = 1, 3a = 2b \\ (V \otimes I_{-2}) / V(-b(a+1)\alpha_3 - (a+1)\alpha_2 \ \ if \ \ , a = 2, b = 1, \ 6d = c \\ V \otimes I_{-2} \ \ otherwise \end{cases}$
- $I_{-4} \cong \begin{cases} (V \otimes I_{-3}) / S^2(I_{-2}), ab > 4, cd > 4 \\ (V \otimes I_{-3}) / V(-(1+c)d\alpha_3 - (1+c)\alpha_1 \oplus S^2(I_{-2}) \ \ if \ \ c = 1, d = 2, \ 6a = b. \\ (V \otimes I_{-3}) / V(-b(a+1)\alpha_3 - (a+1)\alpha_2) \oplus S^2(I_{-2}) \ \ if \ \ , a = 1, b = 2, 3d = 2c \\ V \otimes I_{-2} \ \ \ otherwise \end{cases}$

CONCLUSIONS

In this work, we have considered a particular class of family of quasi hyperbolic Kac Moody algebras QHG_2 and determined the structure of the components in the graded ideals upto level four. This work gives further scope for understanding the structure of the whole algebra and also will further aid in the computation of the multiplicities of roots.

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